Engineering Maths First Aid Kit

8.11

Integration by substitution

Introduction

This technique involves making a substitution in order to simplify an integral before evaluating it. We let a new variable, u say, equal a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience: don't worry if at first you cannot see an appropriate substitution. This skill develops with practice.

1. Making a substitution

Example
Find $\int (3x+5)^6 \mathrm{d}x$.

Solution
First look at the function we are trying to integrate: $(3x+5)^6$. Suppose we introduce a new variable, u, such that $u = 3x+5$. Doing this means that the function we must integrate becomes u^6. This certainly looks a much simpler function to integrate than $(3x+5)^6$. There is a slight complication however. The new function of u must be integrated with respect to u and not with respect to x. This means that we must take care of the term $\mathrm{d}x$ correctly. From the substitution

$$u = 3x + 5$$

note, by differentiation, that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

It follows that we can write

$$\mathrm{d}x = \frac{\mathrm{d}u}{3}$$

The required integral then becomes

$$\int (3x+5)^6 \mathrm{d}x = \int u^6 \frac{\mathrm{d}u}{3}$$

The factor of $\frac{1}{3}$, being a constant, means that we can write

$$\begin{aligned}
\int (3x+5)^6 \mathrm{d}x &= \frac{1}{3}\int u^6 \mathrm{d}u \\
&= \frac{1}{3}\frac{u^7}{7} + c \\
&= \frac{u^7}{21} + c
\end{aligned}$$

To finish off we rewrite this answer in terms of the original variable, x, and replace u by $3x+5$:

$$\int (3x+5)^6 \mathrm{d}x = \frac{(3x+5)^7}{21} + c$$

2. Substitution and definite integrals

If you are dealing with definite integrals (ones with limits of integration) you must be particularly careful with the way you handle the limits. Consider the following example.

Example

Find $\displaystyle\int_2^3 t \sin(t^2) \mathrm{d}t$ by making the substitution $u = t^2$.

Solution

Note that if $u = t^2$ then $\dfrac{\mathrm{d}u}{\mathrm{d}t} = 2t$ so that $\mathrm{d}t = \dfrac{\mathrm{d}u}{2t}$. We find

$$\begin{aligned}\int_{t=2}^{t=3} t \sin(t^2) \mathrm{d}t &= \int_{t=2}^{t=3} t \sin u \, \frac{\mathrm{d}u}{2t} \\ &= \frac{1}{2}\int_{t=2}^{t=3} \sin u \, \mathrm{d}u\end{aligned}$$

An important point to note is that the original limits of integration are limits on the variable t, not u. To emphasise this they have been written explicitly as $t = 2$ and $t = 3$. When we integrate with respect to the variable u, the limits must be written in terms of u too. From the substitution $u = t^2$, note that

$$\text{when } t = 2, u = 4 \quad \text{and when} \quad t = 3, u = 9$$

so the integral becomes

$$\begin{aligned}\frac{1}{2}\int_{u=4}^{u=9} \sin u \, \mathrm{d}u &= \frac{1}{2}\left[-\cos u\right]_4^9 \\ &= \frac{1}{2}(-\cos 9 + \cos 4) \\ &= 0.129\end{aligned}$$

Exercises

1. Use a substitution to find

a) $\int (4x+1)^7 \mathrm{d}x$, b) $\int_1^2 (2x+3)^7 \mathrm{d}x$, c) $\int t^2 \sin(t^3+1) \mathrm{d}t$, d) $\int_0^1 3t^2 \mathrm{e}^{t^3} \mathrm{d}t$.

2. Make a substitution to find the following integrals. Can you deduce a rule for integrating functions of the form $\frac{f'(x)}{f(x)}$?

a) $\int \frac{1}{x+1} \mathrm{d}x$, b) $\int \frac{2x}{x^2+7} \mathrm{d}x$, c) $\int \frac{3x^2}{x^3+17} \mathrm{d}x$.

Answers

1. a) $\frac{(4x+1)^8}{32} + c$, b) 3.3588×10^5 c) $-\frac{\cos(t^3+1)}{3} + c$, d) 1.7183

2. a) $\ln(x+1) + c$, b) $\ln(x^2+7) + c$, c) $\ln(x^3+17) + c$.